Raphael Zentner's instanton class – notes

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If you want to contribute to these notes in any way (e.g., you have spotted a typo), email psuwara at impan dot pl.

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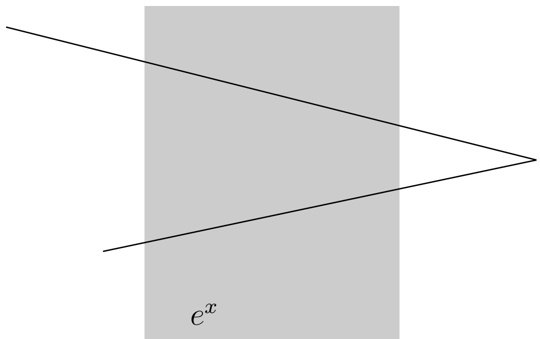

Figure 1: This is a test drawing. I am sure Inkscape will prove useful later on, but for now just consider it a weird piece of art.

1 Lecture 1: 15 IV 2021

1.1 Motivation

Theorem 1.1 ***(Donaldson's Theorem A).*** *If X^4 is a smooth oriented 4-manifold such that the intersection form*

$$Q_X : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

$$Q_X(a,b) = \langle a \cup b, [X] \rangle$$

is negative definite. Then Q_X is equivalent over $\mathbb{Z}$ to the diagonal pairing

$$\begin{aligned} \mathbb{Z}^{b_2(X)} \times \mathbb{Z}^{b_2(X)} &\to \mathbb{Z} \\ (a, \quad b) &\mapsto a^t(-\mathrm{Id})b \end{aligned}$$

In contrast:

Theorem 1.2 ***(Freedman).*** *For any symmetric bilinear unimodular form Q over $\mathbb{Z}$ there exists a topological simply-connected 4-manifold X for which $Q_X \simeq Q$.*

Since there are many negative definite unimodular quadratic forms, we obtain the following:

Corollary 1.3. *There are many topological 4-manifolds which do not admit a smooth structure.*

Other results:

Theorem 1.4 ***(Furuta).*** *Brieskorn homology 3-spheres generate a subgroup $\mathbb{Z}^\infty \subseteq \Theta^3_{\mathbb{Z}}$ of the homology cobordism group.*

Theorem 1.5 ***(Donaldson).*** *The h-cobordism theorem doesn't hold in dimension 4.*

Theorem 1.6 ***(Taubes).*** *There exist infinitely many distinct smooth structures on $\mathbb{R}^4$.*

Note the latter is false for all $\mathbb{R}^n$ for $n \neq 4$!

Theorem 1.7 ***(Kronheimer-Mrowka, Property P).*** *If $K \subseteq S^3$ is a knot and $K \neq U$, U is the unknot, then there exists an irreducible representation $\pi_1\left(S^3_{\frac{p}{q}}\right) \to SU(2)$ if $\left|\frac{p}{q}\right| \leq 2$.*

Theorem 1.8 ***(Zentner).*** *If $Y \neq S^3$ is a closed 3-manifold then there exist non-trivial representations $\pi_1(Y) \to \mathrm{SL}(2, \mathbb{C})$.*

1.2 Fibre bundles

We'll talk about principal fibre bundles, associated vector bundles and connections.

Sources include: Helga Baum: Eichfeld-theorie, Kobayashi-Monizu: Foundations of Differential Geometry.

Definition 1.9 ***(principal fibre bundle).*** Let G be a Lie group. A smooth map $\pi : P \to M$ is called a principal fibre bundle if

- G acts freely on P from the right and is transitive on the fibres,
- π is locally trivial, i.e., for each $x \in M$ there is an open neighborhood $U \ni x$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ such that (here the diagram comes, oh my) commutes and φ is G-equivariant: $\varphi(p) = (\pi(p), h)$ implies $\varphi(pg) = (\pi(p), hg)$.

Exercise 1.10. π admits a global trivialisation if and only if $\pi : P \to M$ admits a section $s : M \to P$ (i.e. $\pi \circ s = \mathrm{id}_M$).

Example 1.11 ***(Hopf bundles).*** $S^{2n+1} \subseteq \mathbb{C}^{k+1}$ with S^1-action by multiplication ($S^1 \subset \mathbb{C}$). Then $\pi : S^{2n+1} \to \mathbb{C}P^n = S^{2n+1}/S^1 = (\mathbb{C}^{k+1} \setminus \{0\})/\mathbb{C}^*$ is a principal S^1-bundle. ♣

Example 1.12 ***(quaternionic Hopf bundles).*** $S^{4n+3} \subseteq \mathbb{H}^{n+1}$, $S^3 \subset \mathbb{H}$ unit spheres. S^3 acts on S^{4n+3} in two different ways, from the right $((q_0, \ldots, q_n), q) \mapsto (q_0 q, \ldots, q_n q)$ or from the left $((q_0, \ldots, q_n), q) \mapsto (\bar{q} q_0, \ldots, \bar{q} q_n)$ (note that for $q \in S^3$ we have $q^{-1} = \bar{q}$).

Then $\pi : S^{4n+3} \to \mathbb{H}P^n$ is a principal S^3-bundle. In particular one gets $S^7 \pi \mathbb{H}P^1 \simeq S^4$. ♣

Example 1.13 ***(frame bundles).*** If $\pi : E \to M$ is a (complex, real, hermitian, euclidean, etc.) vector bundle of rank r, then

$$P_E = \{(e_1, \ldots, e_r) \in E^r | (e_1, \ldots, e_r) \text{is a} \\ \text{(complex, real, unitary, orthogonal, etc.) basis of } E_m = \pi^{-1}(m)\}$$

has a G-action ($\mathrm{GL}(r, \mathbb{C})$, $\mathrm{GL}(r, \mathbb{R})$, $\mathrm{U}(r)$, $\mathrm{O}(r)$, etc.). This forms $\pi : P_E \to M$, a principal G-bundle. The action is given by $(e_1, \ldots, e_r)g = (\sum_{i=1}^r g'_{1i} e_i, \ldots, \sum_{i=1}^r g'_{ri} e_i)$ where $g^{-1} = (g'_{ij})_{i,j=1,\ldots,r}$. ♣

Example 1.14 ***(homogeneous spaces).*** $H \subseteq G$ closed Lie subgroup, G/H is a homogeneous space and $G \to G/H$ is a principal H-bundle. ♣

1.3 Associated bundles

Definition 1.15 ***(associated bundle).*** Let $\pi : P \to M$ be a principal G-bundle. Suppose V is a vector space and $\rho : G \to \mathrm{Aut}(V)$ is a group homomorphism. Then $P \times V$ has a right G-action via $(p, v)g = (pg, \rho(g^{-1})v)$ and $\pi : E = P \times_\rho G = (P \times V)/G \to M$ is the associated bundle to P and ρ.

Exercise 1.16. $\pi : E \to M$ given by $[p, v] \mapsto \pi(p)$ is a vector bundle.

A tautology:

E a G-vector bundle of rank r, $G(E)$ G-frame bundle, then $G(E) \times_G \mathbb{K}^r \to E$ given by $[(e_1, \ldots, e_r), (z_1, \ldots, z_r)] \mapsto \sum z_i e_i$ is an isomorphism of vector bundles.

Further examples:

$$TM = \mathrm{GL}(M) \times_{\rho_{can}} \mathbb{R}^n$$

$$T^*M = \mathrm{GL}(M) \times_{\rho^*_{can}} (\mathbb{R}^n)^*$$

$$\Lambda^k M = \mathrm{GL}(M) \times_{\rho_{can} \wedge \ldots \wedge \rho_{can}} \Lambda^k (\mathbb{R}^n)^*$$

Example 1.17 ***(tautological line bundle over $\mathbb{C}P^n$).***

$$H = \{(l, \xi) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | \xi \in l\}$$

Then

$$H \to L$$
$$(l, \xi) \mapsto l$$

is a complex line bundle. ♣

On the other hand, consider $\pi : S^{2n+1} \to \mathbb{C}P^n$, $\rho_k : S^1 \to \mathrm{Aut}(\mathbb{C})$

$$z \mapsto (\xi \mapsto z^k \xi).$$

Exercise 1.18.

$$H \simeq S^{2k+1} \times_{\rho_1} \mathbb{C}$$

$$H^* \simeq S^{2k+1} \times_{\rho_{-1}} \mathbb{C}$$

$$H^{\otimes l} \simeq S^{2k+1} \times_{\rho_k} \mathbb{C}$$

Denote the Lie algebra of G by $\mathfrak{g}$. Then

$$\begin{aligned} \mathrm{Ad} : G \to & \mathrm{Aut}(G) \\ g \mapsto & (h \mapsto ghg^{-1}) \end{aligned}$$

induces

$$\begin{aligned} \mathrm{ad} : G \to & \mathrm{Aut}(\mathfrak{g}) \\ g \mapsto & (d\,\mathrm{Ad}_g)_e : \mathfrak{g} \to \mathfrak{g} \end{aligned}$$

i.e., take the differential of Ad_g at $e \in G$. Also define

$$\mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g}.$$

By the way,

$$\begin{aligned} (d\,\mathrm{ad})_e : \mathfrak{g} \to & \mathrm{End}(\mathfrak{g}), \\ X \mapsto & (Y \mapsto [X, Y]). \end{aligned}$$

1.4 Connections in principal bundles

Let $\pi : P \to M$ a principal G-bundle. Define

$$VTP = \ker(d\pi : TP \to TM).$$

From the free G-action we get a linear map

$$\begin{aligned} \mathfrak{g} \to & \Gamma(TP) \\ \xi \mapsto & \xi^{\#} \end{aligned}$$

where $\xi^{\#}_p = \frac{d}{dt}\Big|_{t=0} (\rho e^{t\xi})$ using the $e^v = \exp(v)$ the exponential map of G.

Observe that $\xi^{\#} \in \Gamma(VTP)$.

Exercise 1.19. Denote by R_g the right g-action. Then the diagram commutes:

$$\begin{array}{ccc} VTP_p & \xleftarrow[\#]{} & \mathfrak{g} \\ \Big\downarrow{\scriptstyle dR_g} & & \Big\downarrow{\scriptstyle \mathrm{ad}_{g^{-1}}} \\ VTP_{pg} & \xleftarrow[\#]{} & \mathfrak{g} \end{array}$$

Exercise 1.20. $[\xi, \eta]^\# = [\xi^\#, \eta^\#]$

Lemma 1.21. *We get a trivialization* $\# : P \times \mathfrak{g} \to VTP$*.*

Definition 1.22 ***(connection).*** A connection on P is a $\dim M$-dimensional subbundle $H \subset TP$ which is complementary to VTP, i.e., $H \cap VTP = 0$ and $TP = VTP + H$ (shortly $TP = VTP \oplus H$) and equivariant with respect to the G-action, i.e., $dR_g(H) = H$ for any $g \in G$.

Remark 1.23. $d\pi|_H : H \to TM$ is an isomorphism.

Definition 1.24 ***(connection 1-form).*** If H is a connection on $\pi : P \to M$, then we define the associated connection 1-form $\omega_H \in \Omega^1(P; \mathfrak{g})$ by the composition

$$TP_p \xrightarrow{\mathrm{pr}_{||H}} VTP_p \xrightarrow[\simeq]{(\#)^{-1}} \mathfrak{g}$$

Remark 1.25. $\omega_H(X^\#) = X$

Remark 1.26. $R_g^* \omega_H = \mathrm{ad}_{g^{-1}} \omega_H$ by Exercise 1.19 and since H is R_g-invariant.

Remark 1.27. $H = \ker(\omega_H : TP \to \mathfrak{g})$

Lemma 1.28. *Suppose on the other hand that* $\omega \in \Omega^1(P; \mathfrak{g})$ *satisfies* $\omega(X^\#) = X$ *for any* $X \in \mathfrak{g}$ *and* $R_g^* \omega = \mathrm{ad}_{g^{-1}} \omega$ *for any* $g \in G$*. Then* $H_\omega = \ker(\omega : TP \to \mathfrak{g})$ *is a connection.*

Remark 1.29. The two constructions are inverses to each other: $\ker \omega_H = H$ and $\omega_{H_\omega} = \omega$.

Definition 1.30 ***(notation for connections).*** We write A for a connection and H_A or ω_A to make explicit its manifestation.

Definition 1.31 ***(horizontal forms of type ρ).*** Let $\alpha \in \Omega^k(P; V)$ and $\rho : G \to \mathrm{Aut}(V)$. The α is called

- horizontal $\alpha(\xi_1, \ldots, \xi_k) = 0$ whenever any ξ_i is vertical,
- of type ρ if $R_g^* \alpha = \rho(g)^{-1} \circ \alpha$.

Denote horizontal forms of type ρ by

$$\Omega^k_{horiz,\rho}(P; V).$$

Proposition 1.32.

$$\begin{aligned} \Omega^k_{\rho,horiz}(P; V) &\xrightarrow{\sim} \Omega^k(M; P \times_\rho V) \\ \omega &\mapsto \bar{\omega} \end{aligned}$$

is an isomorphism, where

$$\Omega^k(M; P \times_\rho V) = \Gamma(M; \Lambda^k T^* M \otimes P \times_\rho V)$$

and

$$\bar{\omega}_x(v_1, \ldots, v_k) = [p, \omega(\xi_1, \ldots, \xi_k)]$$

where $\pi(p) = x$ and $d\pi_p(\xi_i) = v_i$ for any i.

Remark 1.33. The bracket above does not denote the Lie bracket but the equivalence class of an element in $P \times_\rho V$.

Proof. Independence of lifts: if $d\pi_p(\xi_i) = d\pi_p(\xi_i')$ then $\xi_i - \xi_i' \in VTP$, so by horizontality of ω we get $\omega(\ldots, \xi_i, \ldots) = \omega(\ldots, \xi_i', \ldots)$.

Independence of $p \in \pi^{-1}(x)$ follows since ω is of type ρ. □

Suppose ω_A and $\omega_{A'}$ are two 1-forms. Then

$$\omega_A - \omega_{A'} \in \Omega^1_{\mathrm{ad},horiz}(P; \mathfrak{g})$$

and therefore there exists $a \in \Omega^1_{\mathrm{ad},horiz}(P; \mathfrak{g})$ such that $\omega_{A'} = \omega_A + a$. We conclude that

Lemma 1.34. *The space of connections on $P \to M$ is an affine space over $\Omega^1_{\mathrm{ad},horiz}(P; \mathfrak{g}) \simeq \Omega^1(M; \mathrm{ad}(P))$.*

2 Recitation 1: 20 IV 2021

2.1 Line bundles over the projective space

$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ is an S^1-bundle with action

$$((z_0, \dots, z_n), w) \mapsto (z_0 w, \dots, z_n w)$$

Let

$$\rho_k : \begin{cases} S^1 & \to \mathrm{Aut}(\mathbb{C}) \\ z & \mapsto \mathrm{mult}_{z^k} \end{cases}$$

for $k \in \mathbb{Z}$. Get the associated bundle $P \times_{\rho_k} \mathbb{C} \to \mathbb{C}P^n$, a complex line bundle.

On the other hand we have H as defined in 1.17. The claim is that $S^{2k+1} \times_{\rho_k} \mathbb{C} \simeq H^{\otimes k}$, where H^{-1} is defined as $H^* = \mathrm{Hom}_{\mathbb{C}}(H, \mathbb{C})$.

Starting with ρ_1, we define

$$\begin{array}{ccc} P \times \mathbb{C} & \xrightarrow{f_1} & H \\ \downarrow & \overset{\bar{f}_1}{\nearrow} & \downarrow \\ P \times_{\rho_1} \mathbb{C} & \longrightarrow & \mathbb{C}P^n \end{array}$$

via

$$((z_0, \dots, z_r), w) \xrightarrow{f_1} ([z_0, \dots, z_r], (z_0, \dots, z_r) \cdot w).$$

We directly check it descends to a bundle homomorphism. Since it is an isomorphism on fibers, it is a bundle isomorphism because of the general fact:

Proposition 2.1.

$$\mathrm{GL}(n) \to \mathrm{GL}(n)$$
$$(\mathrm{Aut}(V) \to \mathrm{Aut}(V))$$
$$B \mapsto B^{-1}$$

is a smooth map (polynomial for $\mathrm{U}(n), \mathrm{O}(n), \dots$*).*

Similarly, for $k > 0$ define

$$P \times \mathbb{C} \xrightarrow{f_k} H^{\otimes k},$$
$$((z_0, \dots, z_r), w) \mapsto w \cdot (\underline{z} \otimes \dots \underline{z}).$$

The action of G via $R_g \times \rho_k$ gives $(\underline{z}, w) \simeq (\underline{z} \cdot u, u^{-1}w)$ and f_k descends to the quotient since $(\underline{z}u)^{\otimes k} = u^k \underline{z}$.

We turn to the case $k < 0$. Start with $k = -1$.

$$\begin{aligned} P \times \mathbb{C} &\xrightarrow{f_{-1}} H^* \\ (\underline{z}, w) &\mapsto w \cdot \langle \underline{z}, - \rangle_{\mathbb{C}} \end{aligned}$$

This works because for $u \in S^1$, $\overline{u} \cdot u = 1$.

2.2 Lie bracket exercise

We wanna prove $[X, Y]^{\#} = [X^{\#}, Y^{\#}]$ as well as commutativity of the diagram:

$$\begin{array}{ccc} TP_p & \xleftarrow[\#]{} & \mathfrak{g} \\ \Big\downarrow{\scriptstyle dR_g} & & \Big\downarrow{\scriptstyle \mathrm{ad}_{g^{-1}}} \\ TP_{pg} & \xleftarrow[\#]{} & \mathfrak{g} \end{array} .$$

The commutativity of the diagram is proven this way:

$$\begin{aligned} (\mathrm{ad}_{g^{-1}}(X))^{\#}_{pg} &= \left.\frac{d}{dt}\right|_{t=0} pge^{t\,\mathrm{ad}_{g^{-1}}(X)} \\ &= \left.\frac{d}{dt}\right|_{t=0} pgg^{-1}e^{sX}g \\ &= \left.\frac{d}{ds}\right|_{s=0} pe^{sX}g \\ &= \left.\frac{d}{ds}\right|_{s=0} R_g(pe^{sX}) \\ &= dR_g(X^{\#}_p) \end{aligned}$$

Now recall that on one hand, in the Lie algebra, we have

$$[X, Y] = \left.\frac{d}{ds}\right|_{s=0} \mathrm{ad}_{e^{sX}}(Y)$$

and on a manifold, if ϕ^t_ξ denotes the flow of ξ, then

$$[\xi, \eta](p) = \left.\frac{d}{dt}\right|_{t=0} d\phi^{-t}_\xi \left(\eta_{\phi^t_\xi(p)}\right).$$

We have $G \hookrightarrow P \to M$. Firstly we claim $\phi^t_{X^\#} = R_{e^{tX}}$:

$$\begin{aligned} R_{e^{tX}}(p) &= \left.\frac{d}{ds}\right|_{s=0} R_{e^{(t+s)X}}(p) \\ &= X^\#_{pe^{tX}} = X^\#_{R_{e^{tX}}(p)} \end{aligned}$$

because $R_{e^{(t+s)X}} = R_{e^{sX}} \circ R_{e^{tX}}$.

$$\begin{aligned} [X^\#, Y^\#](p) &= \left.\frac{d}{dt}\right|_{t=0} dR_{e^{-tX}} \left(Y^\#_{e^{tX}(p)}\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} dR_{e^{-tX}} \left.\frac{d}{ds}\right|_{s=0} \left(pe^{tX}\right) e^{sY} \\ &= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} R_{e^{tX}e^{sY}e^{-tX}}(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} R_{\mathrm{ad}_{e^{tX}}(Y)}(p) \\ &= [X, Y]^\#(p) \end{aligned}$$

2.3 Additions to the lecture

Recall that we have $d : \Omega^* N \to \Omega^{*+1} N$ defined by

$$\begin{aligned} d\omega(\xi_0, \ldots, \xi_k) = \sum_{i=0}^{k} (-1)^i \tilde{\xi}_i \omega(\tilde{\xi}_0, \ldots, \hat{\tilde{\xi}}_i, \ldots, \widetilde{\xi}_k) \\ + \sum_{i<j} (-1)^{i+j} \omega([\widetilde{\xi}_i, \widetilde{\xi}_j], \widetilde{\xi}_0, \ldots, \hat{\widetilde{\xi}}_i, \ldots, \hat{\widetilde{\xi}}_j, \ldots, \widetilde{\xi}_k) \end{aligned}$$

where $\widetilde{\xi}_i$ is a vector field with $\tilde{\xi}_i(x) = \xi_i$ (the formula is independent of the choice of $\tilde{\xi}_i$). In particular, for a 1-form we have

$$d\omega(\xi, \eta) = \xi\omega(\eta) - \eta\omega(\xi) - \omega([\xi, \eta]).$$

Notice that d does not in general preserve $\Omega^*_{\rho,horiz}(P, V)$.

Example 2.2. Consider $P = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(t, s) \mapsto t$. Then $\omega = f(s)dt$ is horizontal, but $d\omega = \frac{\partial f}{\partial s} ds \wedge dt$ is not horizontal unless $\frac{\partial f}{\partial s} \equiv 0$. ♣

If we have a connection A on the principal G-bundle $P \to M$ with horizontal subbundle H_A, then define for $\alpha \in \Omega(P, V)$ the differential

$$d_A\alpha = d\alpha \circ \mathrm{pr}_{H_A}$$

i.e., $(d_A\alpha)(\xi_1, \ldots, \xi_k) = d\alpha\left(\mathrm{pr}_{H_A}\xi_1, \ldots, \mathrm{pr}_{H_A}\xi_k\right)$.

Remark 2.3. $d_A\alpha$ is necessarily horizontal, whether or not α has been.

Remark 2.4. If α is of type $\rho : G \to \mathrm{Aut}(V)$, then $d_A\alpha$ is also of this type (because H_A is R_g-invariant).

In particular, get

$$d_A : \Omega^k_{horiz,\rho}(P, V) \to \Omega^{k+1}_{horiz,\rho}(P, V).$$

Definition 2.5. d_A is called the covariant derivative of the connection A on P.

Remark 2.6. $d^2 = 0$, but $d_A \circ d_A \neq 0$ in general.

Definition 2.7. d_A descends to

$$\begin{array}{ccc} \Omega^k_{\rho,horiz}(P;V) & \xrightarrow{d_A} & \Omega^{k+1}_{\rho,horiz}(P;V) \\ \big\downarrow{\scriptstyle -,\simeq} & & \big\downarrow{\scriptstyle -,\simeq} \\ \Omega^k(M; P \times_\rho V) & \overset{\bar{d}_A}{\dashrightarrow} & \Omega^{k+1}(M; P \times_\rho V) \end{array}$$

which we denote by $\bar{d}_A$.

Definition 2.8. Let $E \to M$ be a vector bundle. A map

$$nablaa : \Gamma(E \to M) \to \Omega^1(E \to M) = \Gamma(T^*M \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f \cdot \nabla a$$

for any $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(E)$ is called a covariant derivative.

Proposition 2.9. *Let $P \to M$ be a G-principal bundle and $\rho : G \to \mathrm{Aut}(V)$, then*

$$\overline{d}_A : \Gamma(M; P \times_\rho V) \to \Omega^1(M; P \times_\rho V)$$

is a covariant derivative on the vector bundle $P \times_\rho V$.

Proof. Unravel the definitions. □

Definition 2.10. Let $\gamma : [0,1] \to M$ be a smooth path. We say $s \in \Gamma(E)$ is parallel with respect to ∇ if $(\nabla s)(\dot\gamma(t)) = 0$ for any t. Then $s(\gamma(1))$ is the result of parallel transport of s along γ.

Note that for geodesics we have $\nabla_{\dot\gamma(t)}\dot\gamma(t) \equiv 0$.

On the other hand, having

$$\begin{array}{ccc} G & \hookrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

we would like to take a path in M and lift it to a horizontal path in P, to get parallel transport. We'll do that soon.

3 Lecture 2: 22 IV 2021

Recall we have $d_A = d \circ \mathrm{pr}_A$, that is $d_A\alpha = d\alpha \circ \mathrm{pr}_A$.

3.1 Parallel transport

Consider a bundle

$$\begin{array}{ccc} G & \hookrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

and a path $\gamma : [a,b] \to M$. Suppose A is a connection on P, $u \in \pi^{-1}(\gamma(a)$. Then $\exists! \widetilde{\gamma}_u : [a,b] \to P$ such that $\dot{\widetilde{\gamma}}_u(t) \in (H_A)_{\gamma(t)}$.

Proof: $d\pi_{H_A} : H_A \xrightarrow{\simeq} TM$.

Recall that H_A is a complement of VTP in TP and is R_g-invariant.

Definition 3.1 ***(parallel transport).*** We get a map

$$\mathrm{Par}^A_\gamma : \begin{cases} \pi^{-1}(\gamma(a)) & \to \pi^{-1}(\gamma(b)) \\ u & \mapsto \widetilde{\gamma}_u(b) \end{cases}$$

which is called the parallel transport of γ with respect to A.

Proposition 3.2 ***(properties of parallel transport).*** $\mathrm{Par}^A_{\gamma*\mu} = \mathrm{Par}^A_\mu \circ \mathrm{Par}^A_\gamma$ *and* $\mathrm{Par}^A_\gamma \circ R_g = R_g \circ \mathrm{Par}^A_\gamma$.

Proof. Because H_A is R_g-invariant. □

Exercise 3.3. If Par^A_γ only depends on the endpoint of γ, then the bundle $P \to M$ is trivial and A is the trivial connection ($P \simeq M \times G$, trivial connection is $H = \mathrm{pr}_1^* TM$.

Proof. Hint: define a global section by parallel transport. □

By Proposition 3.2 (properties of parallel transport) Par^A_γ descends to

$$\mathrm{Par}^{E,A}_\gamma : \begin{cases} E_{\gamma(a)} & \to E_{\gamma(b)} \\ [p, v] & \mapsto [\mathrm{Par}^A_\gamma(p), v] \end{cases}$$

on $E = P \times_\rho V$.

Definition 3.4 ***(covariant constancy).*** Suppose ∇ is a covariant derivative on $E \to M$ be a smooth path. A section $s \in \Gamma(E \to M)$ is said to be covariantly constant along γ if

$$(\nabla s)\,(\dot{\gamma}(t)) = 0 \text{ for all } t \in [0, 1].$$

Remark 3.5. This is a differential equation for $\gamma^* s$ on $\gamma^* E \to [0, 1]$.

This also gives a notion of parallel transport

$$\mathrm{Par}_\gamma^\nabla : \begin{cases} E_{\gamma(0)} & \to E_{\gamma(1)} \\ e & \mapsto s(\gamma(1)) \end{cases}$$

if s is covariantly constant along γ and $e = s(\gamma(0))$.

Proposition 3.6. *If ∇_A is a covariant derivative on $E = P \times_\rho V$ coming from a connection A on P, then*

$$\mathrm{Par}_\gamma^{E,A} = \mathrm{Par}_\gamma^{\nabla_A},$$

i.e., the two notions coincide.

Proof.

Given in exercises. Review.

Denote by $\hat{\cdot} : \Gamma(M; P\times_\rho V) \to \Gamma_\rho(P;V)$ and $\bar{\cdot} : \Gamma_\rho(P;V) \to \Gamma(M; P\times_\rho V)$ the isomorphisms...

which

$$\begin{aligned}
\left(\nabla^A_{\dot\gamma(t)} s\right)(\gamma(t)) &= \left(\overline{d}_A s\right)(\dot\gamma(t)) \\
&= d_A \hat{s}\left(\dot{\tilde\gamma}(t)\right) \\
&= \left[\tilde\gamma(t), d_A\hat{s}\left(\dot{\tilde\gamma}(t)\right)\right] \\
&= \left[\tilde\gamma(t), ds\left(\dot{\tilde\gamma}(t)\right)\right] \\
&= \left[\tilde\gamma(t), \frac{d}{dt} s(\tilde\gamma(t))\right] \\
&= \left[\mathrm{Par}_\gamma^A(\tilde\gamma(0)), \frac{d}{dt} s(\tilde\gamma(t))\right] \\
&= \mathrm{Par}_{\gamma_t}^{A,E}\left(\left[\tilde\gamma(0), \frac{d}{dt}\hat{s}(\tilde\gamma(t))\right]\right)
\end{aligned}$$

where $\gamma_t = \gamma|_{[0,t]}$. This is because

Missed.

If $\frac{d}{dt}\hat{s}(\tilde\gamma(t)) = 0$ for any t, then $\hat{s}(\tilde\gamma(1)) = \hat{s}(\tilde\gamma(1))$.

Now $s(\gamma(1)) = \mathrm{Par}_\gamma^{\nabla^A}(s(\gamma(0)))$ if $\nabla^A_{\dot\gamma(t)} s \equiv 0$.
On the other hand

$$\begin{aligned}[\mathrm{Par}_\gamma^A(\tilde\gamma(0)), \hat s(\tilde\gamma(0))] &= \mathrm{Par}_\gamma^{A,E}(s(\gamma(0))) \\ &= [\tilde\gamma(0), \hat s(\tilde\gamma(0))] \\ &= [\tilde\gamma(1), \hat s(\tilde\gamma(0))] \\ &= [\tilde\gamma(1), \hat s(\tilde\gamma(1))] \text{ if } * \text{ holds} \\ &= s(\gamma(1)).\end{aligned}$$

We thus proved that $\mathrm{Par}_\gamma^{A,E}(s(\gamma(0))) = s(\gamma(1))$

What?

□

3.2 Curvature

Definition 3.7. Let $P \xrightarrow{\pi} M$ be a G-principal bundle and A a connection on P. Then $\Omega_A = d_A\omega_A = d\omega_A \circ \mathrm{pr}_{H_A}$ is called the curvature of A.

Remark 3.8. Recall: if X is a manifold and $H \subseteq TX$ is a subbundle, then H is called involutive if $[\eta, \xi] \subseteq H$ for all vector fields $\eta, \xi \in \Gamma(X; H)$.

Theorem 3.9 *(Frobenius)*. *Locally there are submanifolds $Y \subseteq X$ such that $TY = H$ if and only if H is involutive.*

Proposition 3.10. $\Omega_A \equiv 0 \iff H_A$ *is involutive*

Proof. Let $\xi, \eta \in \Gamma(P; H_A)$. Then

$$\begin{aligned}\Omega_A(\xi,\eta) &= d\omega_A(\xi,\eta) \\ &= \xi.\omega_A(\eta) - \eta.\omega_A(\xi) - \omega_A([\xi,\eta]) \\ &= -\omega_A([\xi,\eta]) \\ &\neq 0 \text{ iff } [\xi,\eta] \text{ has a vertical component.}\end{aligned}$$

where we used that $\omega_A(\eta) = \omega_A(\xi) = 0$ by the definition of ω_A.

The last statement in the equation follows since $\omega_A|_{VTP} : VTP \to P \times \mathfrak{g}$ is an isomorphism. □

Proposition 3.11. $R_g^* \Omega_A = \mathrm{ad}_{g^{-1}}$

Proof.

$$\begin{aligned} R_g^* d_A \omega_A &= d\omega_A \circ \mathrm{pr}_{H_A} \circ dR_g \\ &= d\omega_A \circ dR_g \circ \mathrm{pr}_{H_A} \text{since } H_A \text{ is } G\text{-invariant} \\ &= dR_g^* \omega_A \circ \mathrm{pr}_{H_A} \\ &= d\, \mathrm{ad}_{g^{-1}} \omega_A \circ \mathrm{pr}_{H_A} \text{since} R_g^* \omega_A = \mathrm{ad}_{g^{-1}} \\ &= \mathrm{ad}_{g^{-1}} \circ \Omega_A \text{by commutativity of } d \text{ and ad} \end{aligned}$$

□

So $\Omega_A \in \Omega^2_{horiz,\mathrm{ad}}(P; \mathfrak{g})$. Under $\Omega^2_{horiz,\mathrm{ad}}(P; \mathfrak{g}) \simeq \Omega^2(M; \mathrm{ad}(P)) = P \times_{\mathrm{ad}} \mathfrak{g})$ we denote the image by $F_A = \overline{\Omega}_A$.

Proposition 3.12 ***(Cartan's formula).*** *$\Omega_A = d\omega_A + \frac{1}{2}[\omega_A \wedge \omega_A]$, where the latter is a hybrid notation for $[,] \otimes \wedge$.*

Proof. Check for $\Omega_A(\xi, \eta)$.

Say first ξ, η are both vertical vector fields, without loss of generality $\xi = X_p^\#$ and $\eta = Y_p^\#$. Then LHS is identically zero since horizontal. RHS is this.

$$\begin{aligned} &\left(d\omega_A + \frac{1}{2}\ [\omega_A \wedge \omega_A] \right) (X^\#, Y^\#) \\ &\qquad = d\omega_A(X^\#, Y^\#) + \frac{1}{2}[\omega_A(X^\#), \omega_A(Y^\#)] - \frac{1}{2}[\omega_A(Y^\#), \omega_A(X^\#)] \\ &\qquad = X^\# \omega_A(Y^\#) - Y^\# \omega_A(X^\#) - \omega_A([X^\#, Y^\#]) + [\omega_A(X^\#), \omega_A(Y^\#)] \\ &\qquad = -\omega_A([X,Y]^\#) + [X,Y] \text{ since } \omega_A(X^\#) = X \text{ and is constant} \\ &\qquad = -[X,Y] + [X,Y] = 0 \end{aligned}$$

Now let one vector field be horizontal $\widetilde{v}$, that is, G-invariant horizontal lift of $v \in \Gamma(TM)$. Again LHS is zero, and compute the RHS.

$$\begin{aligned} d\omega_A(\widetilde{v}, X^\#) &= \widetilde{v}\omega_A(X^\#) - X^\#\omega_A(\widetilde{v}) - \omega_A([\widetilde{v}, X^\#]) \\ &= \tilde{v}(X) - X^\#(0) - \omega_A([\tilde{v}, X^\#]) = 0 \text{ since} \\ [X^\#, \tilde{v}]_p &= \frac{d}{dt}\bigg|_{t=0} d\left(R_{e^{-tX}}\tilde{v}_{pe^{tX}}\right) \\ &= \frac{d}{dt}\bigg|_{t=0} d\left(\tilde{v}_p\right) \text{ since } \tilde{v} \text{ is a } G\text{-invariant horizontal lift} \\ &= 0 \\ [\omega_A \wedge \omega_A](\tilde{v}, X^\#) &= 0 \text{ since } \tilde{v} \text{ is horizontal.} \end{aligned}$$

Now both are horizontal and R_g-invariant, $\widetilde{v}, \widetilde{w}$. LHS is:

$$\begin{aligned} \Omega_A(\widetilde{v}, \widetilde{w}) &= d\omega_A(\widetilde{v}, \widetilde{w}) \\ &= \widetilde{v}\omega_A(\widetilde{w}) - \widetilde{w}\omega_A(\widetilde{v}) - \omega_A([\widetilde{v}, \widetilde{w}]) \\ &= 0 \text{ since } \omega_A(horizontal) = 0. \end{aligned}$$

RHS is

$$(d\omega_A + \frac{1}{2}[\omega_A \wedge \omega_A])(\tilde{v}, \tilde{w}) = d\omega_A(\tilde{v}, \tilde{w}) + 0.$$

□

Proposition 3.13. *Let $\alpha \in \Omega^1(P; V)$. Then $d_A\alpha = d\alpha + \rho_*(\omega_A) \wedge \alpha$ where $\rho : G \to \mathrm{Aut}(V)$ and $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is its derivative.*

Proof. Just as with Cartan's formula, check on pairs of vertical and horizontal, horizontal and horizontal, vertical and vertical sections. □

Remark 3.14. Also true for $\alpha \in \Omega^k_{\rho,horiz}(P; V)$, where $\left(\rho_*(\omega_A) \wedge \alpha\right)(\xi_0, \dots, \xi_k) = \sum_{i=0}^k (-1)^i \rho_*(\omega_A(\xi_i))\alpha(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_k$.

Recall that for any two connections A, A' there exists a 1-form $a \in \Omega^1_{\rho-eq,horiz}(P; \mathfrak{g})$ such that $\omega_{A'} = \omega_A + a$.

Proposition 3.15. $\Omega_{A+a} = \Omega_A + d_A a + \frac{1}{2}[a \wedge a]$

Proof.

$$\begin{aligned}\Omega_{A+a} &= d\omega_{A+a} + \frac{1}{2}[\omega_{A+a} \wedge \omega_{A+a}] \\ &= d\omega_A + da + \frac{1}{2}[\omega_A \wedge \omega_A] \\ &+ \frac{1}{2}[\omega_A \wedge a] + \frac{1}{2}[a \wedge \omega_A] + \frac{1}{2}[a \wedge a] \\ &= \Omega_A + da + [\omega_A \wedge a] + \frac{1}{2}[a \wedge a]\end{aligned}$$

$$= \Omega_A + d_A a + \frac{1}{2}[a \wedge a]$$

applying the previous proposition to $\rho = \mathrm{ad}$, $\rho_* = [-,]$. □

Proposition 3.16 ***(Bianchi's identity).*** $d_A \Omega_A = 0$

Proof.

$$\begin{aligned}d_A\Omega_A(\xi,\eta,\lambda) &= d\Omega_A(\xi,\eta,\lambda) + [\omega_A \wedge \Omega_A](\xi,\eta,\lambda) \text{ by a Proposition} \\ &= \frac{1}{2}d[\omega_A \wedge \omega_A](\xi,\eta,\lambda) + [\omega_A \wedge d\omega_A](\xi,\eta,\lambda) + [\omega_A \wedge \frac{1}{2}[\omega_A \wedge \omega_A]](\xi,\eta,\lambda) \\ &= \frac{1}{2}[d\omega_A \wedge \omega_A](\ldots) - \frac{1}{2}[\omega_A \wedge d\omega_A](\ldots) + [\omega_A \wedge d\omega_A](\ldots) \\ &+ \frac{1}{2}[\omega_A \wedge [\omega_A \wedge \omega_A](\ldots) \\ &= \frac{1}{2}[\omega_A \wedge [\omega_A \wedge \omega_A]](\xi,\eta,\lambda).\end{aligned}$$

Now without loss of generality $\xi, \eta, \lambda = X^\#, Y^\#, Z^\#$ for $X, Y, Z \in \mathfrak{g}$. So the last term is zero because of the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$. □

Proposition 3.17. *For* $\alpha \in \Omega^k_{horiz,\rho}(P;V)$ *and* A *a connection on* P *we have* $d_A d_A \alpha = \rho_*(\Omega_A) \wedge \alpha$.

Proof.

$$\begin{aligned} d_A d_A \alpha &= d(d\alpha + \rho_*(\omega_A) \wedge \alpha) \\ &+ \rho_*(\omega_A) \wedge (d\alpha + \rho_*(\omega_A) \wedge \alpha) \text{by a Proposition above} \\ &= \rho_*(d\omega_A) \wedge \alpha - \rho_*(\omega_A) \wedge d\alpha \\ &+ \rho_*(\omega_A) \wedge d\alpha + \rho_*(\omega_A) \wedge \rho_*(\omega_A) \wedge \alpha \end{aligned}$$

Now

$$\begin{aligned} \left(\rho_*(\omega_A) \wedge \rho_*(\omega_A)\right)(\xi,\eta) &= (\rho_*(\omega_A)(\xi))\rho_*(\omega_A(\eta)) - \rho_*(\omega_A(\eta))\rho_*(\omega_A(\xi)) \\ &= [\rho_*(\omega_A(\xi)), \rho_*(\omega_A(\eta))] \text{ Lie bracket in End}(V) \\ &= \rho_*([\omega_A(\xi), \omega_A(\eta)]) \text{ since } \rho_* \text{ is a Lie alg. homom.} \\ &= \rho_*\left(\frac{1}{2}[\omega_A \wedge \omega_A]\right)(\xi,\eta) \text{ from Cartan's formula.} \end{aligned}$$

The last step is the following. Write $\omega = \sum X_i \alpha_i$ where $X_i \in \mathfrak{g}$ and $\alpha_i \in \Omega^1(P)$.

$$\begin{aligned} [\overline{\omega}_A \wedge \omega_A](\xi,\eta) &= \sum_{i,j} [X_i, X_j] \alpha_i \wedge \alpha_j(\xi,\eta) \\ &= \sum_{i,j} [X_i, X_j](\alpha_i(\xi)\alpha_j(\eta) - \alpha_i(\eta)\alpha_j(\xi)) \\ &= [\omega_A(\xi), \omega_A(\eta)] - [\omega_A(\eta), \omega_A(\xi)] \\ &= 2[\omega_A(\xi), \omega_A(\eta)] \end{aligned}$$

which finishes the proof. □

Definition 3.18 ***(curvature of a covariant derivative).***
The curvature of a covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is defined by

$$R^\nabla(\xi,\eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}$$

Proposition 3.19. *If A is a connection on $P \to M$ then on $P \times_\rho V$ we get a covariant derivative ∇^A induced from A. Then $R^{\nabla_A} = \rho_*(F_A)$ where $F_A \in \Omega^2(M; \mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g})$ and $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$.*

Proof. Exercise. □

Next time we will consider

$$\begin{array}{ccc} G \circlearrowright P & \overset{f}{\dashrightarrow} & Q \circlearrowleft H \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

which is defined to be a bundle homomorphism for a Lie group homomorphism $\phi : G \to H$ if $f(pg) = f(p)\phi(g)$ for any p, g.

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$\phi : \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$) polynomial of degree k, alternatively $\phi : \mathfrak{g}^k \to \mathbb{R}$ multilinear and invariant under permutations (symmetric).

Suppose ϕ is ad-invariant:

$$\phi(\mathrm{ad}_g X_1, \dots, \mathrm{ad}_g X_k) = \phi(X_1, \dots, X_k)$$

for any $g \in G$ and $X_1, \dots, X_k \in \mathfrak{g}$.

Apply this to $g = e^{tX}$ and differentiate at $t = 0$. Get

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \phi(\mathrm{ad}_{e^{tX}} X_1, \dots, \mathrm{ad}_{e^{tX}} X_k) \qquad (1) \\ &= \phi([X, X_1], X_2, \dots, X_k) + \phi(X_1, [X, X_2], X_3, \dots, X_k) + \dots \end{aligned}$$

Let A be a connection on $P \to M$, define

$$c_\phi(A) = \phi(\Omega_A \wedge \dots \wedge \Omega_A) \in \Omega^{2k}_{horiz}(P)$$

where the curvature Ω_A of A is exterior multiplied $2k$ times.

Two facts from last time: $d_A \Omega_A = 0$ (Bianchi identity) and if $\alpha \in \Omega_{horiz,\rho}(P; V)$ then

$$d_A \alpha = d\alpha + \rho_*(\omega_A) \wedge \alpha \qquad (2)$$

(using a hybrid notation at the end, where $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$).

Proposition 4.1. *$c_\phi(A)$ is closed and for any other connection A' on P the difference $c_\phi(A) - c_\phi(A')$ is exact, hence $[c_\phi(A)] \in H^{2k}(P;\mathbb{R})$ is independent of A.*

Proof.

$$\begin{aligned} dc_\phi(A) &= \phi(d\Omega_A \wedge \Omega_A \wedge \ldots \wedge \Omega_A) + \phi(\Omega_A \wedge d\Omega_A \wedge \Omega_A \wedge \ldots \Omega_A) \\ &= k \cdot \phi(d\Omega_A \wedge \Omega_A \wedge \ldots \Omega_A) \\ &= k \cdot \phi((d\Omega_A + [\omega_A \wedge \Omega_A]) \wedge \Omega_A \wedge \ldots \wedge \Omega_A) \\ &= k \cdot \phi(d_A\Omega_A \wedge \Omega_A \wedge \ldots) \\ &= 0 \text{ by Bianchi identity} \end{aligned}$$

Indeed, (1) implies

$$0 = \phi([\omega_A \wedge \Omega_A] \wedge \Omega_A \wedge \ldots \wedge \Omega_A) + \phi(\Omega_A \wedge [\omega_A \wedge \Omega_A] \wedge \ldots \wedge \Omega_A) + \ldots$$

□

Let A' be another connection, $a = A' - A \in \Omega^1_{horiz,\mathrm{ad}}(P;\mathfrak{g})$. Then $A_t = A + ta$ is a path of connections from A to A'. Then

$$\begin{aligned} \Omega_{A_t} &= \Omega_A + d_A(ta) + \frac{1}{2}t^2[a \wedge a] \\ \Longrightarrow \frac{d}{dt}\Omega_{A_t} &= d_A a + t[a \wedge a] \\ &= d_{A_t} a \end{aligned}$$

Lemma 4.2. *If $B \in \Omega^*_{horiz}(P;\mathbb{R})$ is G-invariant, then $d_A B = dB$.*

Proof. Use (2) with ρ trivial. □

$$\begin{aligned}\frac{d}{dt}c_\phi(A_t) &= k\cdot\phi\left(\frac{d\Omega_{A_t}}{dt}\wedge\Omega_{A_t}\wedge\ldots\wedge\Omega_{A_t}\right)\\ &= k\cdot\phi(d_{A_t}a\wedge\Omega_{A_t}\wedge\ldots\wedge\Omega_{A_t}).\\ &= k\cdot d_{A_t}\phi(a\wedge\Omega_{A_t}\wedge\ldots\wedge\Omega_{A_t}) \text{ by Bianchi}\\ &= k\cdot d\phi(a\wedge\Omega_{A_t}\wedge\ldots\wedge\Omega_{A_t}) \text{ by Lemma}\\ \implies c_\phi(A')-c_\phi(A) &= d\left(k\int_0^1\phi(a\wedge\Omega_{A_t}\wedge\ldots\wedge\Omega_{A_t})\right).\end{aligned}$$

This finishes the proof of the proposition.

Example 4.3. If $\mathfrak{g}$ matrix Lie algebra of a matrix Lie group G, then $\det(t\cdot \mathrm{Id}+X)=\sum_{k=0}t^k\phi_k(X)$ and ϕ_k is an ad_G-invariant polynomial of degree $\mathrm{rk}(G)-k$. ♣

Example 4.4. $G=U(1)$ then $\mathfrak{g}=\mathrm{u}(1)=i\mathbb{R}$. ♣

Remark 4.5. If $P=M\times G$ is the trivial bundle, then it admits the trivial connection pr_1^*TM which has 0 curvature (is integrable).

This implies $[c_\phi]=0$ for any ϕ in this case.

Lemma 4.6. *If $\rho: G\to \mathrm{Aut}(V)$ is trivial, then*

$$\Omega^*(M;P\times_\rho V)\simeq\Omega^*_{horiz,\rho}(P;V)$$

given by π^*.

Notice that $d\pi^*=\pi^*$. Therefore there exists a unique class $\check{c}_\phi(A)\in\Omega(M;\mathbb{R})$ such that $\pi^*\check{c}_\phi(A)=c_\phi(A)$. In fact $\check{c}_\phi(A)=\phi(F_A\wedge\ldots\wedge F_A$ where $F_A\in\Omega^2(M;\mathrm{ad}(P))$.

Example 4.7.

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3\\ & & \downarrow\\ & & S^2\end{array}$$

Hopf fibration given by $\mathbb{C}\ni(z,w)\to[z:w]\in\mathbb{C}P^1$. We will apply the above to $c_\phi(A)=-\frac{1}{2\pi i}\Omega_A\in\Omega^2(S^3;\mathbb{R})$.

The ad-action is trivial for $G=S^1$.

Anyway, what is $[\check{c}_\phi(A)] \in H^2_{\mathrm{dR}}(S^2;\mathbb{R})$? we have the deRham isomorphism $H^2_{\mathrm{dR}}(S^2) \to \mathbb{R}$ given by $[\omega] \mapsto \int_{S^2} \omega$. Chart for $\mathbb{C}P^1$ is

$$\begin{aligned} \phi : \mathbb{C} &\to \mathbb{C}P^1 \\ u &\mapsto [u:1]. \end{aligned}$$

Then $\phi(\mathbb{C}) = \mathbb{C}P^1 \setminus \{[1:0]\}$.

Exercises:

$$\begin{aligned} \omega_A &= \overline{\omega} d\omega + \overline{z} dz \\ \Omega_A &= d\Omega_A \\ &= d\overline{w} \wedge dw + d\overline{z} \wedge dz \\ &= -dw \wedge d\overline{w} - dz \wedge d\overline{z}. \end{aligned}$$

Need to find $F_A \in \Omega^2(S^2; i\mathbb{R})$ such that $\pi^* F_A = \Omega_A$. We will express F_A through the chart f.

We are looking for a section:

$$\begin{array}{c} \pi^{-1}(\phi(\mathbb{C})) \subset S^3 \\ s \uparrow \downarrow \pi \\ \phi(\mathbb{C}) \end{array}$$

and in fact $F_A|_{\phi(CC)} = s^*\Omega_A$ because then $\pi^* F_A = \pi^* s^* \Omega_A = \Omega_A$ because something.

$$\begin{array}{ccc} & \pi^{-1}(\phi(\mathbb{C})) \subset S^3 & \\ & s \uparrow \downarrow \pi & \\ \mathbb{C} \xrightarrow{\phi} & \phi(\mathbb{C}) & \subset S^3 \end{array}$$

A candidate is $s([u:1]) = \frac{(u,1)}{\sqrt{|u|^2+1}}$. Note it is well-defined since $s(p) = \frac{(\phi^{-1}(p),1)}{\sqrt{|\phi^{-1}(p)|^2+1}}$. Now take $F_A = s^*\Omega_A$ and thus $\phi^* F_A = \phi^* s^* \Omega_A = (s \circ \phi)^* \Omega_A$ and $s \circ \phi(u) = \frac{(u,1)}{\sqrt{|u|^2+1}}$. Get

$$(s \circ \phi)^* \Omega_A = -\left(d\left(\frac{u}{\sqrt{|u|^2+1}} \right) \wedge d\left(\frac{\overline{u}}{\sqrt{|u|^2+1}} \right) \right) + 0$$

$$d\left(\frac{u}{\sqrt{|u|^2+1}}\right)=\frac{du}{\sqrt{|u|^2+1}}-\frac{1}{2}u\frac{\overline{u}du+ud\overline{u}}{(|u|^2+1)^{3/2}}$$

and similarly for $\overline{u}$. At the end

$$\begin{aligned}(s\circ\phi)^*\Omega_A &= -\frac{du\wedge d\overline{u}}{|u|^1+1}-\frac{1}{2}\frac{|u|^2}{(|u|^2+1)^2}du\wedge d\overline{u}-\frac{1}{2}\frac{|u|^2}{(|u|^2+1)^2}du\wedge d\overline{u}+0\\ &= -\left(\frac{du\wedge d\overline{u}}{(|u|^1+1)^2}\right)\end{aligned}$$

and thus

$$\begin{aligned}\int_{\mathbb{C}P^1}\left(-\frac{1}{2\pi i}F_A\right) &= \int_{\phi(\mathbb{C})}\left(-\frac{1}{2\pi i}F_A\right)\\ &= \int_{\mathbb{C}}\left(-\frac{1}{2\pi i}\phi^*F_A\right)\\ &= \int_{\mathbb{C}}\frac{1}{2\pi i}\frac{du\wedge d\overline{u}}{(1+|u|^2)^2}\\ &= -\int_{\mathbb{C}}\frac{1}{\pi}\frac{dx\wedge dy}{(1+|u|^2)^2}\\ &= -\frac{1}{\pi}\int_0^{2\pi}\left(\int_0^\infty\frac{rdr}{(1+r^2)^2}\right)d\phi\\ &= -2\cdot\left(\frac{1}{2}\left(\frac{-1}{1+r^2}\right)\right)|_0^\infty\\ &= -1\end{aligned}$$

Conclusion: $-1=[c(\text{Hopf bundle})]\in H^2_{\mathrm{dR}}(\mathbb{C}P^1)$. ♣

4.1 Reduction and extension of the structure group

Let $\lambda: H\to G$ be a Lie group homomorphism. Let $\pi: P\to M$ be a principal G-bundle.

Definition 4.8. A λ-reduction of P is a principal H-bundle $\pi' : Q \to M$ together with a map $f : Q \to P$ satisfying:

- $$\begin{array}{ccc} Q & \xrightarrow{\quad f \quad} & P \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

  is commutative,

- $f(ph) = f(p)\lambda(h)$ for any $p \in Q, h \in H$ (i.e., of type λ).

Example 4.9. $\mathrm{SO}(M) \hookrightarrow Gl(M)$ inclusion of the oriented orthonormal frame bundle is a $\mathrm{SO}(n)$-reduction of the frame bundle of M (exists if TM is orientable). ♣

Remark 4.10. P admits a λ-reduction iff there exists cocycles (g_{ik}) coming from cocycles $h_{ik} : U_i \cap U_k \to H$ such that $g_{ik} = \lambda(h_{ik})$.

Example 4.11.

$$\lambda : S^1 \to S^1$$
$$z \mapsto z^2$$

Claim: the Hopf bundle $S^3 \to S^2$ does not admit a λ-reduction.

Exercise. Use Chern classes later on. ♣

Example 4.12. A $\mathrm{U}(n)$-principal bundle $P \to M$ admits a reduction to a $\mathrm{SU}(n)$-principal bundle iff $P \times_{\det} \mathbb{C}$ is the trivial bundle. ♣

By the way, if we consider the unique connected double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ then a $\mathrm{SO}(n)$-bundle admits a reduction to a $\mathrm{Spin}(n)$-bundle if $w_2(P \times_{can} \mathbb{R}^2) = 0$.

5 Exercise session 4 V 2021

We proved a proposition first.

Then we proved this:

Proposition 5.1. *If $E = P \times_\rho V$, A a connection on P, then*

$$\begin{array}{ccc} \Gamma_G(P;V) & \xrightarrow{\rho_*} & a \\ \downarrow & & \downarrow \\ a & \longrightarrow & a \end{array}$$

Proof.

Missed.

This implies

$$\Big(R_x^{\nabla^A}(v_x, w_x)\varphi\Big)(x) = R_x^{\nabla^A}(v_x, w_x)[p, \hat{\varphi}]$$

$$\begin{aligned} &= [p, (\tilde{v}_p.(\tilde{w}.\hat{\varphi}) - \tilde{w}_p.(\tilde{v}.\hat{\varphi}) - \widetilde{[v,w]}_p.\hat{\varphi})(p)] \text{ by (*)} \\ &= [p, \Big([\tilde{v}, \tilde{w}].\hat{\varphi} - \widetilde{[v,w]}_p.\hat{\varphi}\Big)(p)] \end{aligned}$$

The commutator $[\tilde{v}, \tilde{w}]$ does not need to be horizontal since H_A may not be involutive, but $d\pi([\tilde{v}, \tilde{w}]) = [d\pi(\tilde{v}), d\pi(\tilde{w})] - [v, w] = d\pi(\widetilde{[v,w]})$. So we get

$$\begin{aligned} &= [p, (\Pi_V([\tilde{v}, \tilde{w}]).\hat{\varphi})(p) \\ &= -[p, \omega_A([\tilde{v}, \tilde{w}])_p^{\#}.\hat{\varphi}] \text{ by definition of } \omega_A \\ &= -[p, \Omega_A(\tilde{v}, \tilde{w})^{\#}.\hat{\varphi}] \end{aligned}$$

And note that d_A

Missed.

$$\begin{aligned} \Omega_A(\tilde{v}, \tilde{w})^{\#}.\hat{\varphi}(p) &= \frac{d}{dt}\Big|_{t=0} \varphi\Big(pe^{t\widehat{\Omega_A}(\tilde{v},\tilde{w})}\Big) \\ &= \frac{d}{dt}\Big|_{t=0} \rho\big(e^{-t\Omega_A(\tilde{v},\tilde{w})}\big) \\ &= -\rho_*(\Omega_A(\tilde{v}, \tilde{w}))\hat{\varphi}(p) \end{aligned}$$

And we end with

$$= [p, \rho_*(\Omega_A(\tilde{v}, \tilde{w}))\hat{\varphi}(p)]$$

and thus

$$R_x^{\nabla^A}(v_x, w_x)[p, \hat{\varphi}(p)] = [p, \rho_*(\Omega_A(\tilde{v}, \tilde{w}))\hat{\varphi}(p)]$$

□

Fill in using https://drive.google.com/file/d/1MSc7NIcqWuKAHvGQv4rbwGx6TBQS07No/view.

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Example 6.1. Of a reduction. The tangent bundle to $\mathbb{C}P^2$ admits a reduction with respect to $S^1 \to S^1$, $z \mapsto z^2$, and thus a spin structure. ♣

Definition 6.2. In the above situation (previous lecture), P is called a λ-extension of Q.

Extensions always exist. Indeed, let us take

$$\begin{array}{ccc} H & \hookrightarrow & Q \\ & & \downarrow \\ & & M \end{array}$$

and $\lambda : H \to G$. Then taking $P = Q \times G / H$ where H acts by

$$(h, (q, g)) \mapsto (qh, \lambda(h)g).$$

The right G-action on P is induced from

$$((q, g), g') \mapsto (q, gg').$$

Then $f : Q \to P$ of type λ is given by $f(q) = [q, e]$.

6.1 Reductions, extensions and connections

Proposition 6.3. *Let*

$$\begin{array}{ccc} Q & \xrightarrow{\quad f \quad} & P \\ & {\scriptstyle \pi_Q} \searrow \quad \swarrow {\scriptstyle \pi_P} & \\ & M & \end{array}$$

and f be of type $\lambda : H \to G$. Let A be a connection on Q. Then there is a unique connection A' on P such that $df_q((H_A)_q) = (H_{A'})_{f(q)} \subseteq TP$ for any q. Thus satisfies:

$$f^*\omega_{A'} = \lambda_* \circ \omega_A, \tag{3}$$
$$f^*\Omega_{A'} = \lambda_* \circ \Omega_A. \tag{4}$$

Here $\lambda_* : \mathfrak{g} \to \mathfrak{h}$ is the associated Lie algebra homomorphism.

Proof. Let $p = f(q)g$. Define

$$(H_{A'})_p = dR_g\Big(df_q\Big((H_A)_q\Big)\Big).$$

Check that this is independent of q with the property $\pi_P(p) = \pi_Q(q)$ (this uses the H-invariance of H_A and the fact that f is of type λ.

The $H_{A'}$ then are clearly G-invariant and they form a complement:

$$\begin{aligned} & \pi_P \circ f = \pi_Q \\ \implies \quad & (d\pi_P)_{f(q)} \circ df_q|_{H_A} \xrightarrow{\cong} TM_{\pi_Q(q)} \end{aligned}$$

so $df_q\Big((H_A)_q\Big)$ is a complement to $VTP_{f(q)} = \ker(d\pi_P)_{f(q)}$. So $H_{A'}$ is a connection. Uniqueness follows from the required G-invariance.

Consider $X \in \mathfrak{h}$.

$$(f^*\omega_{A'})(X^{\#}) = \omega_{A'}(df(X^{\#})) \text{ by definition.}$$

$$
\begin{aligned}
df_q(X^{\#}) &= \frac{d}{dt}\bigg|_{t=0} f(qe^{tX}) \\
&= \frac{d}{dt}\bigg|_{t=0} f(q)\lambda\big(e^{tX}\big) \\
&= \frac{d}{dt}\bigg|_{t=0} f(q)e^{t\lambda_* X} \\
&= (\lambda_* X)^{\#}_{f(q)} \\
&= \omega_A \cdot \Big((\lambda_*(X))^{\#}\Big) \\
&= \lambda_*(X)
\end{aligned}
$$

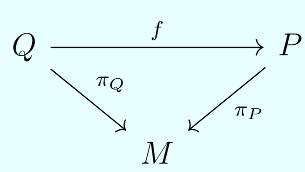
This proves the first formula, (3). The second formula, (4), follows from this and

$$
\begin{aligned}
\Omega_{A'} &= d\omega_{A'} + \frac{1}{2}[\omega_{A'} \wedge \omega_{A'}] \\
\implies f^*\Omega_{A'} &= df^*\omega_{A'} + \frac{1}{2}[f^*\omega_{A'} \wedge f^*\omega_{A'}] \\
&= d\lambda_*\omega_A + \frac{1}{2}[\lambda_*\omega_A \wedge \lambda_*\omega_A] \\
&= \lambda_*\Omega_A \text{ because } \lambda_* \text{ is a Lie alg. hom.}
\end{aligned}
$$

□

Definition 6.4. A' is called the λ-extension of A and A the λ-reduction of A'.

Proposition 6.5. *Let*



be a morphism of type $\lambda : H \to G$ such that λ_* is a Lie algebra isomorphism. Suppose Ais a connection on P. then there exists a unique connection A' denoted by f^*A such that $f^*\omega_A = \lambda_*\omega_{A'}$.

Proof. Define
$$\omega_{A'} = \lambda_*^{-1} \circ f^*\omega_A \in \Omega^*(Q; \mathfrak{h}).$$
□

For instance, this applies to
$$\mathrm{Spin}(n) \xrightarrow{2:1} \mathrm{SO}(n)$$
$$\mathrm{Spin}^c(n) \xrightarrow{2:1} \mathrm{SO}(n) \times S^1$$
both inducing Lie algebra isomorphisms.

Definition 6.6 ***(bundle isomorphism).*** A bundle homomorphism f of type id_G is called a bundle isomorphism. In particular, $f : P \to P$ of type id_G is called a bundle automorphism.

Definition 6.7 ***(gauge group).*** $\mathrm{Aut}(P) = \{f : P \to P \text{ of type } \mathrm{id}_G\}$ is called the gauge group of P.

Remark 6.8.
$$\begin{aligned}\mathrm{Aut}(P) &\simeq \mathcal{C}^\infty_G(P; G)\\ &= \left\{\varphi : P \to G \middle| \varphi(pg) = \mathrm{Ad}_g \varphi(p) (= g^{-1}\varphi(p)g) \forall_{p,g}\right\}\end{aligned}$$
Indeed, given φ we set $f(p) = p\varphi(p)$ and get $f(pg) = pg\varphi(pg) = pgg^{-1}\varphi(p)g = f(p)g$.

In other words,
$$\mathrm{Aut}(P) \simeq \Gamma(M; \mathrm{Ad}(P) = P \times_{\mathrm{Ad}} G).$$
Note $\mathrm{Ad}(P)$ is not a principal G-bundle.

Proposition 6.9. *Let $f \in \mathrm{Aut}(P)$ and $\varphi_f : P \to G$ be the associated map. Then f^*A (defined by $\omega_{f^*A} = f^*\omega_A$) satisfies:*

1. $\omega_{f^*A} = \mathrm{Ad}_{\varphi^{-1}} \omega_A + \varphi^{-1} d\varphi$ *(the last one is left multiplication in the Lie group),*
2. $d_{f^*A} = f^* \circ d_A \circ (f^*)^{-1}$, $\Omega_{f^*A} = \mathrm{Ad}_{\varphi_f^{-1}} \circ \Omega_A$.

Proof. Exercise. (Baum, Thm 3.22) □

6.2 Chern-Weil theory again

$$\begin{array}{ccc} G & \hookrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

principal bundle. Consider $\phi : \mathfrak{g}^k \to \mathbb{C}$ which is Ad-invariant and symmetric. Get

$$c_\phi(A) = \phi(\Omega_A^{\wedge k}) \in \Omega^{2k}_{horiz,G-invt}(P;\mathbb{C}).$$

We have seen $dc_\phi(A) = 0$ and $c_\phi(A') = c_\phi(A) \in d\Omega^*_{horiz,G-invt}(P;\mathbb{R})$.

Lemma 6.10. $\Omega^*_{horiz,G-invt}(P;\mathbb{C}) \simeq \Omega^*(M;\mathbb{C})$ *where the inverse map is given by* π^*.

Thus there is $\overline{c}_\phi(A)$ such that $\pi^*\overline{c}_\phi(A) = c_\phi(A)$, and we define

$$c_\phi(P) = [\overline{c}_\phi(A)] \in H^{2k}(M;\mathbb{C}).$$

Definition 6.11 ***(Weil homomorphism).*** We get the Weil homomorphism

$$\begin{aligned} W_P : S^*_G(\mathfrak{g}) &\longrightarrow H^*_{dR}(M;\mathbb{C}) \\ \phi &\longmapsto c_\phi(P) \end{aligned}$$

defined on the algebra of symmetric multilinear forms.

Definition 6.12. Given $f : N \to M$, we call $f^*P = \{(n,p) \in N \times P | f(n) = \pi(p)\}$ the pull-back bundle, and get

$$\begin{array}{ccc} f^*P & \xrightarrow{\hat{f}(n,p)=p} & P \\ \downarrow & & \downarrow \pi \\ N & \xrightarrow{f} & M \end{array}.$$

Proposition 6.13. *If A is a connection on P, then there is a unique connection A' on f^*P such that $\omega_{A'} = \hat{f}^*\omega_A$.*

Proof. Define it by this formula, $\omega_{A'} = \hat{f}^*\omega_A$. Check:

$$\begin{aligned} \omega_{A'}(X^{\#}) &= \omega_A((\hat{f}_* X^{\#}) \\ &= \omega_A(X^{\#}) = X. \end{aligned}$$

□

Theorem 6.14 ***(functoriality of the Weil homomorphism).*** *For the Weil homomorphism we have*

1. *$c_\phi(f^*P) = f^*c_\phi(P)$ (or, $W_{f^*P} = f^*W_P$),*
2. *if*

$$\begin{array}{ccc} Q & \xrightarrow{\;f\;} & P \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

and ω of type λ, $\phi \in S^*_G(\mathfrak{g})$ and $\phi_\lambda = \phi \circ \lambda_* \in S^*_G(\mathfrak{h})$, then $c_{\phi_\lambda}(Q) = c_\phi(P)$ (i.e., $W_P(\phi) \circ \lambda_* = W_Q(\phi_\lambda)$).

Proof. (1) follows from the previous proposition:

$$\begin{aligned} c_\phi(f^*P) &= \left[\bar{c}_\phi\left(\hat{f}^*A\right)\right] \\ &= [f^*c_\phi(A)] \\ &= f^*c_\phi(P) \end{aligned}$$

(2) as well, use the above proposition with the push-forward connection and notice f induces id on M. □

Remark 6.15. It can be shown that any principal G-bundle admits a reduction to a maximal compact subgroup (e.g., $\mathrm{U}(n) \subseteq \mathrm{GL}(n, \mathbb{C})$).

6.3 Chern classes of vector bundles

$$\mathrm{U}(n) = \{B \in \mathrm{GL}(n,\mathbb{C}) | B^*B = \mathrm{id}\}$$
$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) | X^* = -X\}$$

Definition 6.16. $\phi_k : \mathfrak{gl}(n,\mathbb{C}) \to \mathbb{C}$ is given by

$$\det\Big(t - \frac{1}{2\pi i}X)\Big) = \sum_{k=0}^{n} \phi_k(X) t^{n-k}.$$

Then ϕ_k are Ad-invariant and $\phi_k|_{\mathfrak{u}(n)}$ take real values. (proof: $\overline{\det\big(t - \frac{1}{2\pi i}X\big)} = \det\Big(t + \frac{1}{2\pi i}\overline{X}^t\Big) = \det\big(t - \frac{1}{2\pi i}X\big)$).

Any $X \in \mathfrak{u}(n)$ is diagonalizable, $X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Thus

$$\det\Big(t - \frac{1}{2\pi i}X\Big) = \prod_{i=1}^{n}\Big(t - \frac{\lambda_i}{2\pi i}\Big)$$

which implies that

$$\phi_1(X) = -\frac{1}{2\pi i}\operatorname{tr}(X)$$

and

$$\begin{aligned}
\phi_2(X) &= \sum_{i<j} \lambda_i\lambda_j\Big(-\frac{1}{4\pi^2}\Big) \\
&= -\frac{1}{8\pi^2}\Bigg(\sum_{i,j}\lambda_i\lambda_j - \sum_i \lambda_i^2\Bigg) \\
&= \frac{1}{8\pi^2}(\operatorname{tr}(X^2) - \operatorname{tr}(X)\operatorname{tr}(X))
\end{aligned}$$

and so on, to

$$\phi_n(X) = \Big(-\frac{1}{2\pi i}\Big)^n \det(X).$$

Theorem 6.17. *The elements* $\phi_0, \ldots, \phi_n \in \mathrm{Sym}^*_{U(n)}(\mathfrak{u}(n))$ *are algebraically independent and generate* $\mathrm{Sym}^*_{U(n)}(\mathfrak{u}(n))$.

Proof. $\phi_k(X) = \left(-\frac{1}{2\pi i}\right)\sigma_k(\lambda_1, \ldots, \lambda_n)$, where σ_k is the k-th elementary symmetric polynomial. These symmetric polynomials have the required property. □

Theorem 6.18. *Let $E = P \times_{\rho_{can}} \mathbb{C}^n$, where P is a principal $\mathrm{U}(n)$-bundle. Then*

$$c_k(E) = [\phi_k(F_A \wedge \ldots \wedge F_A))] \in H^{2k}_{dR}(M; \mathbb{R})$$

is the image of the Chern class under

$$H^{2k}(X; \mathbb{Z}) \to H^{2k}(X; \mathbb{R}) \simeq H^{2k}_{dR}(X).$$

The element

$$\begin{aligned} c(E) &= \left[\det\left(1 - \frac{1}{2\pi i} F_A\right)\right] \\ &= c_0(E) + c_1(E) + \ldots c_n(E) \end{aligned}$$

is called the total Chern class.

Proof. Sketch. Chern classes are characterized by the axioms:

- $c_1(E_1) = c_1(E_2)$ if $E_1 \simeq E_2$,
- $c(f^*E) = f^*c(E)$ for any $f : N \to M$,
- $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$, (cup product)
- $c_k(E^*) = (-1)^k c_k(E)$, $c(\underline{\mathbb{C}}^n) = 1$,
- $\langle c_1(H), [\mathbb{C}P^1]\rangle = -1$ where $H \to \mathbb{C}P^1$ is the tautological line bundle.

The first two follow from the functoriality of the Weil homomorphism, Theorem 6.14.

When $E_1 \oplus E_2$ has a reduction to a $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C})$-bundle, take a connection respecting this reduction and apply Theorem 6.14.

E^* is associated to the dual representation $\mathrm{U}(n) \to (\mathbb{C}^n)^*$, $(\rho_{can})^* : \mathrm{u}(n) \to \mathrm{gl}(n, \mathbb{C})$ mapping $X \mapsto X^* = -X$.

We have verified $c_1(H) = -1$ last time. □

Example 6.19. $c_1(E) = -\frac{1}{2\pi i}[\mathrm{tr}(F_A)]$

$c_2(E) = \frac{1}{8\pi^2}[\mathrm{tr}(F_A \wedge F_A) - \mathrm{tr}(F_A) \wedge \mathrm{tr}(F_A)]$ ♣

Remark 6.20. If we have a reduction to $\mathrm{SU}(n)$, then $c_1(E) = 0$.

Proof. $\phi_1|_{\mathrm{su}(n)} = -\frac{1}{2\pi i}\,\mathrm{tr}\,|_{\mathrm{su}(n)} = 0$ since elements in $\mathrm{su}(n)$ are traceless. □

6.4 Pontryagin classes

Definition 6.21 ***(Pontryagin classes).*** If $E \to M$ is a real vector bundle, then $p_k(E) = (-1)^k c_{2k}(E^{\mathbb{C}})$ is a Pontryagin class.

The total Pontryagin class is $p(E) = 1 + p_1(E) + \ldots \in H^{4*}_{dR}(M; \mathbb{R})$.

Theorem 6.22. *These can be obtained by the above approach from* $\det\left(t - \frac{1}{2\pi}X\right) = \sum_{k=0}^{n} \psi_k(X) t^{n-k}$ *on* $\mathfrak{gl}(n)$ *and* $\psi_{2k+1}|_{\mathfrak{o}(n)} = 0$ *for any* k*, where* $\mathfrak{o}(n)$ *is the Lie algebra of* $\mathrm{O}(n) = \{A | AA^t = \mathrm{id}\}$*, so* $\mathfrak{o}(n) = \{X | X^t = -X\}$*.*

For instance $\mathrm{tr}(X) = \mathrm{tr}(X^t) = \mathrm{tr}(-X) = -\mathrm{tr}(X)$ *which implies* $\mathrm{tr}(X) = 0$*.*

Random notes

Remind Raphael about recording if needed.

Raphael's lecture notes are available at https://drive.google.com/file/d/10F8GW2ad0rY9Y0QlUyJbJ1s_GGP9Nasb/view?usp=sharing.